

**$Q^2$ -DEPENDENCE OF DEEP INELASTIC SUM RULES\***

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In this talk, I will concentrate on  $Q^2$ -dependence of deep inelastic sum rules. I will first give a modern definition of deep-inelastic sum rules and then discuss physical origins of their scaling violation at finite  $Q^2$ . Following this, I discuss a few well-known examples, in particular, the Bjorken sum rule, which is at the center of interest of this symposium. Finally, I consider the  $Q^2 \rightarrow 0$  limit of sum rules using low-energy theorems. I think this can motivate some interesting CEBAF physics.

**1. Deep-Inelastic Sum Rules**

Let me start with the structure functions of the nucleon. In inclusive lepton-nucleon scattering, one measures the following hadron tensor,

$$W_{\mu\nu} = \frac{1}{4\pi} \int e^{iq \cdot \xi} d^4\xi \langle PS | J_\mu^\dagger(\xi) J_\nu(0) | PS \rangle , \quad (1)$$

where  $J_\mu$  is the electroweak current of quarks,  $S$  and  $P$  are polarization and momentum vectors of the nucleon, respectively, and  $q$  is the momentum of a virtual boson.  $W_{\mu\nu}$  can be decomposed into various scalar structure functions of the nucleon<sup>1</sup>, which are shown in Table 1. The different column shows dependence on the nucleon polarization: Unpolarized, longitudinally-polarized, and transversely polarized. The parity-even structure functions are measurable in electromagnetic processes, whereas the parity-odd ones couple through weak interactions. The structure functions with circles are the ones that have been measured in previous experiments. The cross or check under each structure function labels the beam polarization (no or yes) when it is measured.

The structure functions depend on two Lorentz scalars,  $Q^2 = -q^2$  and  $\nu = p \cdot q$ . In the deep-inelastic limit, *i.e.*,  $Q^2, \nu \rightarrow \infty$ ,  $Q^2/2\nu = x = \text{fixed}$ , the structure functions scale to quark distributions, which are only functions of  $x$ , (neglecting renormalizations point dependence for the moment),

$$\begin{aligned} W_2 &\rightarrow F_2 \sim q(x) + \bar{q}(x) , \\ W_3 &\rightarrow F_3 \sim q(x) - \bar{q}(x) , \\ G_1 &\rightarrow g_1 \sim \Delta q(x) + \Delta \bar{q}(x) , \\ X_1 &\rightarrow a_1 \sim \Delta q(x) - \Delta \bar{q}(x) , \end{aligned} \quad (2)$$

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**Table I:** Structure functions

	Un-polarized	L-polarized	T-polarized
$P$ -even	$\textcircled{W_2}, \textcircled{W_L}$	$\textcircled{G_1}$	$G_2$
(Beam-pol)	$\times$	$\checkmark$	$\checkmark$
$P$ -odd	$\textcircled{W_3}$	$X_L, X_2$	$Y_1$
(Beam-pol)	$\checkmark$	$\times$	$\times$

where quark distribution  $q(x)$  and quark helicity distribution  $\Delta q(x)$  are defined as,

$$\begin{aligned}
q(x) &= \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P | \bar{\psi}(0) \not{n} \psi(\lambda n) | P \rangle , \\
\Delta q(x) &= \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS | \bar{\psi}(0) \not{n} \gamma_5 \psi(\lambda n) | PS \rangle ,
\end{aligned} \tag{3}$$

where  $n$  is a light-like vector. Two things can be said about these distributions. First, they are light-cone correlations as the quark fields in the matrix elements are separated along the light-cone. Second, they are related to the ground state properties of the nucleon. Because of this second property, one can immediately derive the structure function sum rules at  $Q^2 = \infty$  (the deep inelastic sum rules),

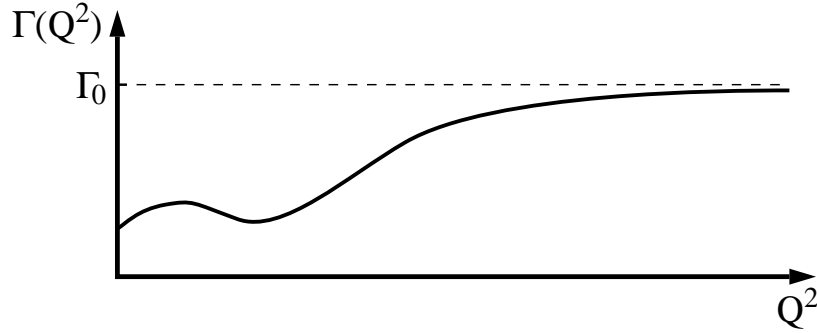
$$\begin{aligned}
\int_0^1 F_3^{\nu p + \bar{\nu} p}(x) dx &= 3 , \\
\int_0^1 g_1^{ep - en}(x) dx &= \frac{1}{6} g_A , \\
\frac{18}{5} \int_0^1 F_2^{ep + en}(x) dx &= \Delta p .
\end{aligned} \tag{4}$$

The first is the Gross-Llewellyn Smith sum rule, the second is the Bjorken sum rule, and the third is the momentum sum rule with  $\Delta p$  denoting the momentum fraction of the nucleon carried by quarks.

## 2. Scaling Violation At Finite $Q^2$

At finite  $Q^2$ , the deep-inelastic sum rules are violated by quark-gluon interactions. Thus one can write a generalized sum rule,

$$\int_0^1 (\dots) dx = \Gamma(Q^2) , \tag{5}$$



**Figure 1.** Schematic  $Q^2$ -dependence of a deep inelastic sum rule.

where  $\Gamma(Q^2)$  is an unknown function of  $Q^2$ , except at the  $Q^2 \rightarrow \infty$  limit, where it approaches the deep-inelastic limit  $\Gamma_0$ . A schematic drawing for  $\Gamma(Q^2)$  is shown in Fig. 1.

At present, there is no general theory about  $Q^2$  variation of  $\Gamma(Q^2)$ . However, for  $Q^2 > Q_0^2$ , some scale related to hadron masses or non-perturbative physics, we believe the following twist expansion is correct,

$$\Gamma(Q^2) = E_0 \left( \frac{Q^2}{\mu^2} \right) \Gamma_0(\mu^2) + E_2 \left( \frac{Q^2}{\mu^2} \right) \Gamma_2(\mu^2)/Q^2 + E_4 \left( \frac{Q^2}{\mu^2} \right) \Gamma_4(\mu^2)/Q^4 + \dots, \quad (6)$$

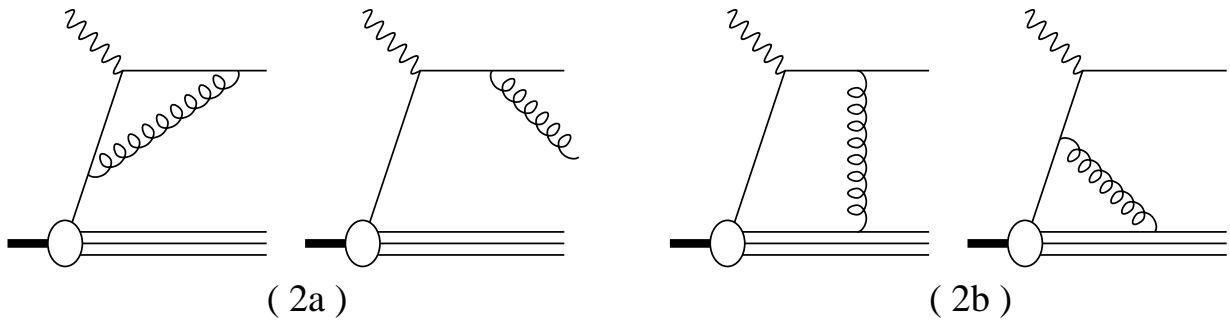
where the coefficient functions have expansions in the strong coupling constant,

$$E_n = \sum_{i=0}^{\infty} \alpha_s^i(Q^2) e_n^i. \quad (7)$$

The terms beyond the first in Eq. (6) are suppressed by successive powers of  $1/Q^2$  and are called higher twist corrections.  $\Gamma_n(\mu^2)$  are related to the nucleon matrix elements of *local* operators containing quark and gluon fields,

$$\Gamma_n(\mu^2) = \langle PS | \hat{O}_n(\mu^2) | PS \rangle. \quad (8)$$

It is clear, then, that  $Q^2$  dependence of  $\Gamma(Q^2)$  is introduced through both power-dependence (higher twists) and logarithmic dependence in the strong coupling constant (QCD radiative corrections). Both contributions can be represented by Feynman diagrams as shown in Fig. 2.



**Figure 2.** Feynman diagrams giving  $Q^2$ -dependence to deep inelastic sum rules: (2a) radiative corrections, (2b) higher-twist effects.

It must be emphasized, however, the above picture is not complete. Recently, works were produced which go beyond this canonical understanding. Some examples: Balitsky and Braun have considered instanton contributions to deep-inelastic structure functions, which cannot simply be classified as power or log corrections<sup>2</sup>; Muller has considered non-perturbative improvement to the perturbative series in (7) and has shown that it gives rise to power-like terms which are not included in (6) [Ref. 3]. In the subsequent discussion, I assume these contributions are small.

### 3. Examples

In the following discussion, I show three examples of twist expansion: the Bjorken sum rule, the Gross-Llewellyn Smith sum rule, and the  $F_L(x)$  sum rule.

With  $Q^2$ -dependent corrections, the Bjorken sum rule can be written as,

$$\int_0^1 g_1^{p-n}(x) dx = \frac{g_A}{6} \left( 1 - \frac{\alpha_S}{\pi} - 3.58 \left( \frac{\alpha_S}{\pi} \right)^2 - 20.2 \left( \frac{\alpha_S}{\pi} \right)^3 + \dots \right) + \frac{\mu_4^{p-n}}{Q^2} + \dots \quad (9)$$

where QCD radiative corrections to the twist-two contribution have been calculated by Larin *et al.*<sup>4</sup>. The first non-trivial power corrections were first calculated by Shuryak and Vainshtein<sup>5</sup>, and have recently been checked by Ji and Unrau<sup>6</sup>. The result for  $\mu_4$  is,

$$\mu_4 = \frac{M^2}{9} \sum_f e_f^2 (a_{2f} + 4d_{2f} - 4f_f) \quad , \quad (10)$$

where  $a_{2f}$ ,  $d_{2f}$ , and  $f_f$  are related to the matrix elements of twist-two, three, and four operators. These higher-twist matrix elements have been evaluated in terms of the QCD sum rule method and Bag model. (I will neglect the QCD radiative corrections associated with these matrix elements).

Let me consider these corrections at  $Q^2 = 2 \text{ GeV}^2$ . To evaluate the QCD radiative corrections, I take  $\Lambda_{\overline{MS}}^{(4)} = 260_{-46}^{+54} \text{ MeV}$  from the particle data book, which corresponds to  $\alpha_S(M_Z) = 0.1134 \pm 0.0035$ , a world-average including LEP data and deep inelastic scattering fits. The error is enlarged to 10% at  $Q^2 = 2 \text{ GeV}^2$ ,

$$\alpha_S(2 \text{ GeV}^2) = 0.330 \pm 0.035 \quad . \quad (11)$$

This gives the number in the bracket in (9)  $0.835_{-0.031}^{+0.023}$ , representing a 17% correction. The correction from still higher power of  $\alpha_S$  terms is about  $20 \left( \frac{\alpha_S}{\pi} \right) \sim 0.03$ , a 3% correction. Thus, neglecting the higher twists, the Bjorken sum rule at  $Q^2 = 2 \text{ GeV}^2$  is

$$\int_0^1 g_1^{p-n}(x, Q^2) dx = 0.175 \pm 0.008 \quad . \quad (12)$$

The higher twist matrix elements have been evaluated in the Bag model by Ji and Unrau<sup>6</sup>, who found,

$$\frac{M_4^{p-n}}{Q^2} = 0.031 \frac{M^2}{Q^2} = 0.014 \quad , \quad (13)$$

which is an 8% correction. On the other hand, the same matrix elements, when evaluated in QCD sum rule (Balitsky, Braun, Kolesnichenko<sup>7</sup>), give,

$$\frac{M_4^{p-n}}{Q^2} = -0.023 \frac{M^2}{Q^2} = -0.011 \quad , \quad (14)$$

which is a 6% correction with an opposite sign to (13). Given this large uncertainty on the matrix elements, I give a  $\pm 0.014$  correction to the Bjorken sum rule from the higher twists. Thus, the best knowledge for the sum rule from the theoretical side is,

$$\int_0^1 g_1^{p-n}(x) dx = 0.175 \pm 0.008 \pm 0.014 \quad . \quad (15)$$

This is consistent with the present experimental determination of this sum rule from E142 experiment<sup>8</sup>.

On the other hand, if one considers the Bjorken sum rule at  $Q^2 = 10 \text{ GeV}^2$ , the theoretical uncertainty is much smaller, in fact,

$$\int_0^1 g_1^{p-n}(x) dx = 0.188 \pm 0.003 \pm 0.003 \quad , \quad (16)$$

at  $Q^2 = 10 \text{ GeV}^2$ .

The deviation from Gross-Llewellyn smith sum rule is usually characterized by  $\Delta$  defined as,

$$\Delta = 1 - \frac{1}{3} I_{\text{GLS}} \quad , \quad (17)$$

$$I_{\text{GLS}} = \int_0^1 F_3^{\nu p + \bar{\nu} p}(x) dx \quad .$$

Experimentally,  $\Delta$  has been measured by CCFR collaboration at  $Q^2 = 3 \text{ GeV}^2$  [Ref. 9],

$$\Delta^{\text{exp}} = 0.167 \pm 0.006 \pm 0.026 \quad . \quad (18)$$

Theoretically,  $\Delta$  can be expressed as

$$\Delta^{\text{th}} = Q \frac{\alpha_S}{\pi} + 3.58 \left( \frac{\alpha_S}{\pi} \right)^2 + 19.0 \left( \frac{\alpha_S}{\pi} \right)^3 + \cdots + \frac{8}{27} \frac{d}{Q^2} + \frac{\text{T.M.}}{Q^2} + \cdots \quad , \quad (19)$$

where  $d$  is the matrix element of a twist-four operator and T.M. means target mass correction. The twist-four matrix element has been calculated in the Bag model and in the QCD sum rule, both giving a consistent result,  $0.33 \text{ GeV}^2$ . Substituting it to (19), I have

$$\Delta^{\text{th}} = 0.170 + \frac{\text{T.M.}}{Q^2} \quad . \quad (20)$$

Without the target mass correction,  $\Delta^{\text{th}}$  agrees with  $\Delta^{\text{exp}}$  perfectly. However, when it is included, we have,  $\Delta^{\text{th}} = 0.140$ , which is still within the experimental error bar. Thus, a better precision is needed for  $\Delta^{\text{exp}}$  to discern a non-trivial higher twist effect.

Finally, I consider the  $1/Q^2$  correction to  $F_L(x)$ . It is known that there are contributions from target masses and QCD radiative corrections to  $F_L(x)$ . However, these contributions are not enough to explain the data on  $F_L(x)$ . The residue must come from the non-trivial higher twist contributions. Recently, Choi *et al.*<sup>10</sup> have extracted a moment of the residual  $F_L(x)$  (I call  $\Delta F_L(x)$ ) from the SLAC-MIT, BCDMS, and NMC data,

$$2 \int_0^1 x \Delta F_L(x) dx = \begin{cases} (0.035 \pm 0.012)/Q^2 & (\text{proton}) \\ (0.023 \pm 0.008)/Q^2 & (\text{neutron}) \end{cases} \quad (21)$$

I have calculated the higher twist contribution from the Bag model, the result is,

$$2 \int_0^1 x \Delta F_L(x) dx = \begin{cases} 0.020/Q^2 & (\text{proton}) \\ 0.013/Q^2 & (\text{neutron}) \end{cases} \quad (22)$$

which is roughly consistent with the data. Notice that the data shows a remarkable  $SU(6)$  structure when a ratio is made between the proton and neutron results, which I think is a strong support for the Bag calculation.

#### 4. Sum Rules At $Q^2 \rightarrow 0$

Finally, I discuss the  $Q^2 \rightarrow 0$  limit of the sum rules. I will take the Bjorken sum rule as an example although the discussion can be extended to other sum rules. I introduce,

$$\begin{aligned} \Gamma(Q^2) &= \int_0^1 g_1(x, Q^2) dx \\ &= \frac{Q^2}{2M^2} \int_{Q^2/2}^{\infty} G_1(\nu, Q^2) \frac{d\nu}{\nu} \\ &= \frac{Q^2}{2M^2} I_1(Q^2) \end{aligned} \quad (23)$$

It was observed by Anselmino *et al.*<sup>11</sup> that  $I_1(0) = -\kappa^2/4$  from Drell-Hearn-Gerasimov sum rule, where  $\kappa$  is the anomalous magnetic moment. Since  $I_1(Q^2)$  is positive at large  $Q^2$ , they argue that higher twists must be unusually large so as to make the variation of  $I_1(Q^2)$  smooth. I will show below that this is not the case.

What they have overlooked is the nucleon's elastic contribution to  $g_1(x, Q^2)$ , which is non-analytic as function of  $Q^2$ . At  $Q^2 = 0$ , the elastic contribution vanishes because of energy momentum conservation. At  $Q^2 \neq 0$ , however small it may be, the elastic contribution to  $g_1(x, Q^2)$  exists. To look at the overall  $Q^2$  dependence of  $\Gamma(Q^2)$  to draw conclusions about the higher twists, one must not neglect this elastic

contribution at low  $Q^2$ . In duality language, which is implicitly assumed here, it is the sum of elastic plus resonance contributions that duals the deep-inelastic twist expansion. According to these arguments, I have shown at low  $Q^2$  [Ref. 12] that,

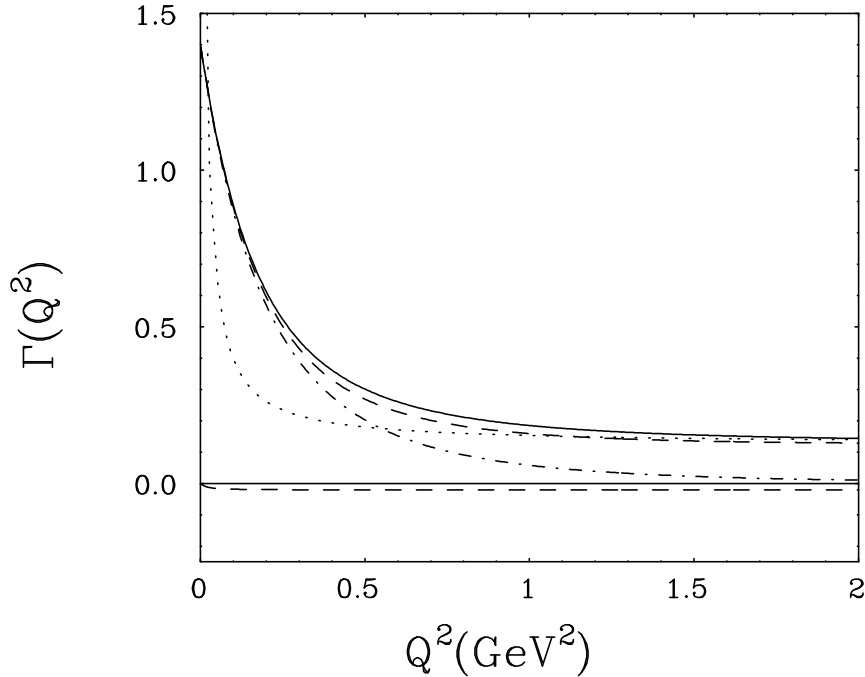
$$\Gamma(Q^2) = \frac{1}{2}F_1(F_1 + F_2) - \frac{F_2^2}{8M^2}Q^2 \quad (24)$$

where  $F_1$  and  $F_2$  are Dirac and Pauli form factors. The first term in (24) comes from elastic contribution and the second from inelastic contributions as summed by Drell-Hearn-Gerasimov sum rule. Clearly, if the nucleon is a point-like object, we have  $\Gamma_1(Q^2) = \frac{1}{2}$  at all  $Q^2$ .

Eq. (24) tells us both  $\Gamma(Q^2)$  and its first derivative at  $Q^2 = 0$ . In fact,

$$\Gamma(0) = \begin{cases} 1.396 & \text{(proton)} \\ 0 & \text{(neutron)} \end{cases} \quad (25)$$

$$\left. \frac{d\Gamma}{dQ^2} \right|_{Q^2=0} = \begin{cases} -8.631\text{GeV}^{-2} & \text{(proton)} \\ -0.479\text{GeV}^{-2} & \text{(neutron)} \end{cases} \quad (26)$$



**Figure 3.** A model for the sum rule  $\Gamma(Q^2)$  at all  $Q^2$ . The solid and upper-dashed curves are the parameterization with the bag and QSR higher twist matrix elements, respectively. The dotted curve represents the result of the twist expansion to order  $1/Q^2$  and the dot-dashed curve represents the elastic contribution. A similar interpolation for the neutron is shown as the lower-dashed curve.

Using these, and the twist-expansion at large  $Q^2$ , we can construct an interpolating sum rule,

$$\Gamma(Q^2) = \frac{1}{2}F_1(F_1 + F_2) \left(1 - \lambda_1 \frac{Q^2}{M^2}\right) + \lambda_2 \frac{1 + \lambda_3 M^2/Q^2}{1 + \lambda_4 M^4/Q^4} \quad (27)$$

which is shown in Fig. 3 for two different choices of higher twist matrix elements. As is clearly seen from the figure, the higher twists of sizes from the Bag or QCD sum rule calculations are consistent with low  $Q^2$  behavior due to the large negative derivative of  $\Gamma$  at the origin.

Finally, let me comment that future experimental data on  $\Gamma(Q^2)$  from low  $Q^2$  (say 0.5 GeV<sup>2</sup>) will be very useful for confirming the picture presented above. The data for the large  $x(> 0.1)$  region may be obtained from CEBAF where resonance physics is important.

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